PMM U.S.S.R.,Vol.51,No.5,pp.579-586,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

CONSEQUENCES OF A NON-INTEGRABLE PERTURBATION OF THE INTEGRABLE CONSTRAINTS: MODEL PROBLEMS OF LOW DIMENSIONALITY*

YA.V. TATARINOV

Mechanical systems with kinematic constraints containing a small parameter are considered. It is assumed that when the value of the parameter is zero, the constraints of such a system become integrable, i.e., a family of holonomic systems is obtained depending on several arbitrary integration constants. Then the methods of perturbation theory can be used to represent, to a first approximation, the motion of the system with non-zero parameter values, as a combination of the motion of a slightly modified holonomic system with slowly varying previous constants (a transgression).

At present, two basic theories exist of the theory of motion of systems with integrable constraints, and both theories have received a validation which took into account the possible physical nature of these constraints (in the case when they are linear and homogeneous with respect to the velocity, and independent of time).

The classical theory (CT) /1-3/: a single idea of Caratheodory was developed in /4/ and refined by Fufayev in /5/, and the author showed in /6/ that the CT systems can be regarded as the limit certain holonomic systems in which dissipative forces of a special type exist (such as strong viscous friction in the region of contact between two bodies). The prospects of studying the transgression within the wide framework of this approach are well illustrated by the following class of problems, namely by that of a body rolling along a fixed surface, with a spherical surface of contact of small radius. If the radius tends to zero, then in the limit a problem of the rotation of a rigid body with a fixed point is obtained.

The variationally axiomatic (called in /7/ the vaxonomic) theory (VT): the motions are by definition the extrema of the action functional from amongst the curves satisfying the constraints (the forces here are assumed to be potential). Hertz in /1/ called such extrema, while studying motion without active forces, the geodesic (nowadays the term has a different meaning and its wide interpretation refers to the theory of connectivities), and did not see any physical meaning in them.

It was, however, shown in /7/ that VT systems are obtained from holonomic systems also by a passage to the limit, but a passage of a different kind, by a change in the metric (not unlike a strongly elliptic tensor of attached masses in the case of a plate in an ideal fluid). It is also possible to combine the CT and VT /8/.

The present paper deals with a group of model problems with constraints of the Chaplygin sledge-type, where both theories can be used. By describing the transgression we can compare the effects of the CT and VT on the general basis of an unperturbed family of Hamiltonian system.

1. The equations of motion in Lagrangian form.

The difference between the CT and VT systems can be clearly observed in almost Chaplygin-type systems. Let the equations of constraints

$$x_s^{\cdot} = f_s(x_1^{\cdot}, \ldots, x_m^{\cdot}, x_1, \ldots, x_m, t),\ s = m+1, \ldots, n \tag{1.1}$$

be given, and the kinetic energy T of the system be independent of $x_{m+1}, \ldots, x_n$, (although the potential energy V may be dependent). We denote by T^* the result of substituting expressions (1.1) into T, and write the equations

$$\frac{d}{dt}\frac{\partial T^*}{\partial x_\lambda^{\cdot}} - \frac{\partial T^*}{\partial x_\lambda} + \frac{\partial V}{\partial x_\lambda} + \sum_s \frac{\partial V}{\partial x_s}\frac{\partial f_s}{\partial x_\lambda^{\cdot}} = \sum_s p_s\left(\frac{d}{dt}\frac{\partial f_s}{\partial x_\lambda^{\cdot}} - \frac{\partial f_s}{\partial x_\lambda}\right) \tag{1.2}$$

To close the system, we must add to (1.1) and (1.2) another $n-m$ equations. These equations will be

$$p_s = \frac{\partial T}{\partial x_s^{\cdot}}\ \text{(CT)};\quad p_s^{\cdot} = -\frac{\partial V}{\partial x_s},\quad p_s(t_0) = 0\ \text{(VT)} \tag{1.3}$$

*Prikl.Matem.Mekhan.,51,5,741-749,1987

(When the initial value of p_s shown for VT is chosen, the solution $x(t)$, $t_0 \leqslant t \leqslant t_1$ will represent a strong extremal of the action functional $\int L dt$ in the sense that its left-hand end does not remain fixed during the variation, but is displaced together with the constraints). Generalization to systems of general form is simple.

If the constraint equations depend on the parameter ε and can be integrated when $\varepsilon = 0$, then we can assume without loss of generality that the equations (1.1) are written in the form

$$x_s^{\cdot} = \varepsilon f_s(x_1^{\cdot}, \ldots, x_m^{\cdot}, x_1, \ldots, x_m, t, \varepsilon) \tag{1.4}$$

When $\varepsilon = 0$, we obtain a family of holonomic systems depending on the constants x_s. When $\varepsilon \neq 0$, we must replace f_s in (1.2) by εf_s; and this gives the transgression equations.

The proposed approach is aimed at bringing closer together the qualitative theories of motion of holonomic and non-holonomic systems. It is clear that there is little chance of solving the transgression equations directly, and to study the solution we shall have to use asymptotic methods, and before anything else, the averaging method. But this will be possible only when we obtain, at $\varepsilon = 0$, a family of integrable Hamiltonian systems (integrable in the sense that there is a sufficient number of integrals in the involution) in which we can use the "action-angle" variables. In the cases when the right-hand sides of the equations of motion and their first derivatives are bounded, the averaged system can be regarded as one adequately reflecting the real mechanical effects of the transgression, using the fact that most solutions of the averaged system are near to the solutions of the initial system over extremely long time periods (see e.g. the results of Bogolyubov and Bes'yes /9/, and Anosov and Neishtadt /10/).

2. An almost holonomic pendulum. A weightless plate moves in the Oxy plane. The plate carries two blades forming the letter T, and the transverse element moves slowly along itself; a point mass M is attached to the plate along the line of the longitudinal blade. Thus the coordinates of the instantaneous centre of velocities C of the plate in the $M\xi\eta$ coordinate system attached to the plate, $\xi = \varepsilon t + \xi_0$, $\eta = r > 0$. Assuming that the force $F = -Mge_y$ acts on the point M we obtain, when $\varepsilon = 0$, a normal pendulum.

If $C(\xi(t), \eta(t))$ is an arbitrary point moving relative to the body according to a known law, then we can use, as the coordinates describing the position of the body, the coordinates x, y of this point in the fixed coordinate system, and the angle φ or rotation of the body. Using such coordinates constructed for the point $(\varepsilon t + \xi_0, r)$ (i.e. x and y are now the coordinates of the centre of velocities of the body), we obtain the following expression for the constraint equation and kinetic energy of the body:

$$dx/d\xi = \cos\varphi, \quad dy/d\xi = \sin\varphi, \quad \xi = \varepsilon t + \xi_0 \tag{2.1}$$

$$2T^* = M(\xi^2 + \eta^2)\varphi^{\cdot 2}$$

Since the potential energy

$$V = Mg(y - r\cos\varphi - \xi\sin\varphi)$$

Eqs.(1.2) become (we shall first assume that the dimensionally independent parameters $M = g = r = 1$)

$$(1 + \xi^2)\varphi^{\cdot\cdot} + 2\varepsilon\xi\varphi^{\cdot} + \sin\varphi - \xi\cos\varphi = \varepsilon(p_x\sin\varphi - p_y\cos\varphi) \tag{2.2}$$

Knowing the Cartesian coordinates of the centre of mass

$$x - \xi\cos\varphi + \sin\varphi, \quad y - \xi\sin\varphi - \cos\varphi$$

we can easily see that in the CT the identities (1.3), taking the constraints into account, are

$$p_x = 0 + (\xi\sin\varphi + \cos\varphi)\varphi^{\cdot}, \quad p_y = 0 + (-\xi\cos\varphi + \sin\varphi)\varphi^{\cdot}$$

while in VT (1.3) can be trivially integrated, and this yields

$$p_x = 0, \quad p_y = -t$$

After substituting into (2.2) we find that the transgression equations in the problem consist of (2.1) and

$$(1 + \xi^2)\varphi^{\cdot\cdot} + \varepsilon N\xi\varphi^{\cdot} + \sin\varphi - N\xi\cos\varphi = 0 \tag{2.3}$$

and $N = 1$ for CT, and $N = 2$ for VT. We shall show, for comparison, that in the case when the pendulum is suspended from the point $C(\xi, 1)$ non-stationary with respect to the body only (i.e. $x^{\cdot} = y^{\cdot} = 0$), the equations of motion are

$$(1 + \xi^2)\varphi^{\cdot\cdot} + 2\varepsilon\xi\varphi^{\cdot} + \sin\varphi - \xi\cos\varphi = 0$$

Here and henceforth terms of the order of ε^2 are neglected without any special discussion.

After the change of variable

$$\varphi = \operatorname{arctg} N\xi + \psi \tag{2.4}$$

Eq.(2.3) will become

$$(1 + \xi^2)\psi^{\cdot\cdot} + \varepsilon N\xi\psi^{\cdot} + \sqrt{1 + (N\xi)^2}\sin\psi = 0 \tag{2.5}$$

From this it follows that the unperturbed system ($\varepsilon = 0$) can be regarded as a pendulum of variable length (the point of suspension moves along $M\eta$)

$$l(\xi) = (1 + \xi^2)(1 + (N\xi)^2)^{-1/2} \tag{2.6}$$

In addition to ξ, x, y, the specific energy (per unit length) of the unperturbed system

$$h = {}^1/_2 l(\xi)\psi^{\cdot 2} + (1 - \cos\psi) \tag{2.7}$$

is also a slow variable ($h = 0$ in the state of equilibrium).

By virtue of (2.5) we have

$$\frac{dh}{d\xi} = -\frac{\xi}{1+\xi^2}\left(1 + \frac{N-1}{1+4\xi^2}\right)(h - (1 - \cos\psi))$$

(the variable ξ plays the part of a slow variable), and the remaining equations are obtained from (2.1) by making the substitution (2.4).

Let us average the equations for x', y', h' over the period of the unperturbed motion /9, 10/ in the oscillatory mode. In the system with energy (2.7),

$$\langle\sin\psi\rangle = 0, \ \langle\cos\psi\rangle = f(h) = 2\mathbf{E}(\sqrt{h/2})/\mathbf{K}(\sqrt{h/2}) - 1$$

where $\mathbf{E}, \mathbf{K}$ are complete elliptic integrals. Therefore the averaged transgression equations are

$$h' = -\frac{\xi}{1+\xi^2}\left(1 + \frac{N-1}{1+4\xi^2}\right)(h + f(h) - 1) \tag{2.8}$$

$$x' = \frac{1}{\sqrt{1+(N\xi)^2}}f(h), \quad y' = \frac{N\xi}{\sqrt{1+(N\xi)^2}}f(h)$$

The function $\mathbf{E}(k)/\mathbf{K}(k)$ decreases monotonically as $k \in [0, 1)$ increases, from 1 to 0. Its derivative increases at the same time without limit, but not faster than $k(1 - k^2)^{-1/2}$. Therefore, when $h \in [0, 2)$ increases, the function $f(h)$ varies from $+1$ to -1, but its derivative is unbounded, its estimation given by the inequality $|f'(h)| \leqslant 2^{1/2}(2 - h)^{-1/2}$. The solutions (2.8) are not uniformly close to the exact solutions. However, by virtue of the averaged system $h' < 0$ and from the estimates given in Sect.3, it follows that the error of the averaging method increases, depending on the initial value (over a constant time interval of the order of $1/\varepsilon$), in accordance with the law $\exp(2 - h_0)^{-1/2}$.

Let us expand f in a Taylor series: $f(h) = 1 - h(1 + h/16)/2 + O(h^3)$. The formula can be obtained using the well-known expansions of $\mathbf{E}, \mathbf{K}$, or by direct computation of $\langle V\rangle$ by expanding V in a Taylor series as in the proof of the Lindschtedt formula /11/.

When the oscillations are small, $h \approx 0$ and

$$x = N^{-1}\operatorname{Arsh} N\xi, \ y = N^{-1}\sqrt{1 + (N\xi)^2}; \ y = N^{-1}\operatorname{ch} Nx \tag{2.9}$$

so that the centre of velocities is displaced along a catenary.

In the case of low energy ($h^2 \approx 0$) oscillation we have

$$h = h_0[l(\xi)]^{-1/N} \tag{2.10}$$

and here the centre of velocities is displaced more slowly. The investigation can be continued along similar lines, since the variables in the equation for h can be separated.

Let s be the natural parameter of the curve $x(\xi), y(\xi)$, and θ the angle between the tangent and the Ox axis; ρ is the radius of curvature. Then $ds/d\xi = |f(h)|$ and

$$\theta = \operatorname{arctg} N\xi, \ \rho = N^{-1}(1 + (N\xi)^2)\,|f(h)| \tag{2.11}$$

Since $h' < 0$, $f' < 0$, ρ, increases all the time provided that $f(h_0) > 0$.

It must be stressed that the brevity of the answer in the case of VT is largely governed by the choice of initial conditions for p_x, p_y, which removed the multivalued dependence of the solutions on the initial conditions characteristic for CT /7/.

Bearing this in mind, we can nevertheless conclude that the CT and VT effects in the almost holonomic pendulum are complex. The line of suspension of the pendulum rotates (by an angle $\operatorname{arctg} N\xi$, just as in the state of rest), the centre of oscillations is displaced non-trivially in the perpendicular direction, and the energy of oscillation, generally speaking, decreases.

3. Some averaging estimates.

Let

$$\chi(\Phi, \varphi) = \chi_0(\Phi) + \chi_*(\Phi, \varphi), \quad \chi_0 = \frac{1}{2\pi}\int_0^{2\pi} \chi\, d\varphi$$

represent the usual separation of a mean value of the vector or scalar function χ, 2π-periodic in φ, dependent also on the variables Φ. We shall also define the vector χ' or the matrix $\partial\chi/\partial\Phi$. Let $|\cdot|$ be the modulus of a scalar, norm of a vector, or the corresponding norm of a matrix.

Let us consider the following three systems of differential equations:

$$\frac{d\Phi}{dt} = \varepsilon F(\Phi, \varphi), \quad \frac{d\varphi}{dt} = \omega(\Phi) + \varepsilon f(\Phi, \varphi) \tag{3.1}$$

$$\frac{d\Theta}{dt} = \varepsilon F_0(\Theta) + \varepsilon^2 R(\Theta, \theta, \varepsilon), \quad \frac{d\theta}{dt} = \omega(\Theta) + \varepsilon f_0(\Theta) + \varepsilon^2 r(\Theta, \theta, \varepsilon) \tag{3.2}$$

$$\frac{d\Psi}{dt} = \varepsilon F_0(\Psi), \quad \frac{d\psi}{dt} = \omega(\Psi) + \varepsilon f_0(\Psi) \tag{3.3}$$

Let the first system be defined in the domain $\mathbf{D}\{\Phi_1, \ldots, \Phi_k\} \times S^1\{\varphi \bmod 2\pi\}$. The third system is obtained from it by averaging over φ (such averaging, which also encompasses the equation for the rapid variable, is interchangeable with any time changes of the form $ds = K(\Phi)dt$). System (3.3) can be obtained in two steps: first we use the mapping L_ε

$$\Theta = \Phi + \varepsilon\Lambda(\Phi, \varphi), \quad \theta = \varphi + \varepsilon\lambda(\Phi, \varphi) \tag{3.4}$$

to transform system (3.1) and (3.2), and we then reject terms of the order of ε^2.

We shall seek Λ, λ with zero means (then the mapping L_ε will not, in the mean, displace or rotate the layers $\Phi = \text{const}$). Writing

$$\Lambda(\Phi, \varphi) = -\left(\int_0^{\varphi} \frac{1}{\omega(\Phi)} F_*(\Phi, \varphi)\, d\varphi\right)_* \tag{3.5}$$

we obtain (λ can be arbitrary)

$$\varepsilon^2 R(\Theta, \theta, \varepsilon) = -\varepsilon\left(F_0(\Phi + \varepsilon\Lambda) - F_0(\Phi)\right) + \varepsilon^2\left(\Lambda' F - (F_*/\omega) f\right) \tag{3.6}$$

System (3.2) is defined, generally speaking, in a narrower domain (depending on ε and such that the mapping L_ε is invertible and does not take the object outside the limits of $\mathbf{D} \times S^1$).

We shall investigate the following solutions of the systems introduced: $\Phi_\varepsilon(t)$, $\varphi_\varepsilon(t)$; $\Psi(\varepsilon t)$, $\psi_\varepsilon(t)$; $\Theta_\varepsilon(t)$, $\theta_\varepsilon(t)$, and here we have

$$\Phi_\varepsilon(0) = \Psi(0) = \mathrm{A}, \quad \varphi_\varepsilon(0) = \psi_\varepsilon(0) = \alpha$$
$$\Theta_\varepsilon(0) = \mathrm{B}_\varepsilon = \mathrm{A} + \varepsilon\Lambda(\mathrm{A}, \alpha), \quad \theta_\varepsilon(0) = \beta_\varepsilon = \alpha + \varepsilon\lambda(\mathrm{A}, \alpha)$$

Let us assumed that the trajectory $\{\Psi(\varepsilon t),\ 0 \leqslant \varepsilon t \leqslant T\}$ lies in D together with its δ-neighbourhood (which will be denoted by Ψ^δ) and let $\max\chi$ denote the maximum value of the modulus or norm χ in this neighbourhood.

From the usual theorems on estimates we have

$$|\Theta_\varepsilon = \Psi| \leqslant |B_\varepsilon - A| \exp(\varepsilon \max F_0' \cdot t) + (\max R/\max F_0')(\exp(\varepsilon \max F_0' \cdot t) - 1)$$

provided that system (3.2) is defined in Ψ^δ and as long as $\Theta_\varepsilon(t) \in \Psi^\delta$. By virtue of (3.4) we have

$$|\Theta_\varepsilon - \Psi_\varepsilon| \leqslant \varepsilon \max\Lambda$$

as long as $\Phi_\varepsilon \in \Psi^\delta$, and $B_\varepsilon - A$ is also estimated accurately. As a result we have

$$|\Phi_\varepsilon - \Psi| \leqslant \varepsilon\left[m(\exp \varepsilon Dt + 1) + D^{-1} \max R(\exp \varepsilon Dt - 1)\right]$$

as long as Φ_ε, $\Theta_\varepsilon \in \Psi^\delta$. Here it was assumed that

$$\max F_0' \leqslant D \tag{3.7}$$

and the estimate $\max\Lambda \leqslant m$ follows from the requirement that

$$\max(F_*/\omega) \leqslant m/\pi \tag{3.8}$$

We have, by virtue of (3.6) (irrespective of λ)

$$\max R \leqslant Dm + \max \Lambda' \max F + (m/\pi) \max f$$

but this inequality will already hold in the region $L_\varepsilon [\Psi^\delta] \cap \Psi^\delta$. In order to arrive at this region, we shall confine ourselves to the $(\delta - \varepsilon m)$-neighbourhood of $\Psi(\varepsilon t)$.

Let us write, together with (3.7) and (3.8):

$$\max (F_*/\omega)' \leqslant km/\pi,\ \max F \leqslant M,\ \max f \leqslant \mu M \tag{3.9}$$
$$c = (k + \mu/\pi)M$$

This yields

$$|\Phi_\varepsilon - \Psi| \leqslant \varepsilon m\,[2e^{\varepsilon D} + cD^{-1}(e^{\varepsilon D} - 1)] \tag{3.10}$$

as long as $\Theta_\varepsilon \in \Psi^{\delta-\varepsilon m}$, $\Phi_\varepsilon \in \Psi^\delta$, and under the assumption that the mapping L_ε is one-to-one. The validity of the latter can be guaranteed by making the neighbourhood a little smaller.

The simplest way of obtaining the required estimates is to put $\lambda \equiv 0$. We note that if the point $\Phi_0 \in \Psi^{\delta-\varepsilon m}$, then the mapping L_ε is one-to-one in its εm-neighbourhood $U^{\varepsilon m}(\Phi_0)$, if $\varepsilon m < 1/k$. Indeed, from

$$\Phi_1 + \varepsilon\Lambda(\Phi_1, \varphi) = \Phi_2 + \varepsilon\Lambda(\Phi_2, \varphi)$$

it follows that

$$|\Phi_1 - \Phi_2| \leqslant \varepsilon \max \Lambda' \cdot |\Phi_1 - \Phi_2| \leqslant \varepsilon km\,|\Phi_1 - \Phi_2| < |\Phi_1 - \Phi_2|$$

Let now $\Phi_1, \Phi_2 \subset \Psi^{\delta-2\varepsilon m}$. If $|\Phi_1 - \Phi_2| < 2m\varepsilon$, then $\Phi_i \subset U^{m\varepsilon}(\Phi_0)$, $\Phi_0 \in \Psi^{\delta-m\varepsilon}$ and therefore $\Theta_1 \neq \Theta_2$. When $|\Phi_1 - \Phi_2| \geqslant 2m\varepsilon$, we obtain the same result simply from $|\Lambda| < m\varepsilon$. Thus the mapping L_ε is one-to-one on $\Psi^{\delta-2m\varepsilon}$ and $L_\varepsilon[\Psi^{\delta-2m\varepsilon}] \subset \Psi^{\delta-m\varepsilon}$. Therefore, if the estimate (3.10) holds for $t = 0$, the condition $\Phi_\varepsilon \in \Psi^{\delta-2m\varepsilon}$ will not be violated as long as the right-hand side of (3.10) remains smaller than $\delta - 2m\varepsilon$ and the condition (3.10) holds during all this time.

We can now assert that for $0 \leqslant \varepsilon t \leqslant T$ we will have

$$|\Phi_\varepsilon - \Psi| < \varepsilon m(2 + c\varepsilon t)e^{\varepsilon Dt} \tag{3.11}$$

as long as the right-hand side is less than $\delta - 2m\varepsilon$. In order to increase the clarity, the estimate (3.10) was slightly relaxed by introducing the inequality

$$(e^{\varepsilon Dt} - 1)/D < \varepsilon t e^{\varepsilon Dt}\ (t > 0) \tag{3.12}$$

Therefore we have, in the same time interval,

$$|\Phi_\varepsilon - \Psi| < \varepsilon C,\ C = m(2 + cT)e^{DT} \tag{3.13}$$

when $\varepsilon < \varepsilon_0 = \delta/(C + 2m)$.

In order to obtain system (3.3) in its entirety, we take together with (3.5),

$$\lambda = -\left(\int_0^\varphi \frac{1}{\omega}[f_* - \omega'\Lambda]\,d\varphi\right)_*$$

If $\omega' \neq 0$, then λ will depend essentially on Λ, and adding to Λ some function $\Lambda_0(\Phi)$ we can, without changing $F_0(\Theta)$, add to $f_0(\Theta)$ any function $g(\Theta)$. Finally, in this case the estimates yield a finite divergence of θ_ε and ψ over time periods of the order of $1/\varepsilon$. The only possibility of obtaining estimates of the order of ε involves passing to a new time $ds = \omega^{-1}dt$ and not returing to the old time.

For this reason we shall simply assume that $\omega = \text{const}$; and now $\max \lambda \leqslant \pi\,\omega^{-1} \max f_*$.

Let us ensure that L_ε is one-to-one when $\lambda = 0$. To do this we shall impose the demands already derived above for L_ε with $\lambda = 0$. Then we can write

$$\Phi = \Theta + \varepsilon K(\Theta, \varphi, \varepsilon)$$

and substitute this into λ. It now remains to ensure that the function φ, ε with parameter Θ: is monotonic

$$\theta = \varphi + \varepsilon\lambda(\Theta + \varepsilon K(\Theta, \varphi, \varepsilon), \varphi)$$

i.e. to require that

$$\frac{\varepsilon}{\pi\omega}\left[\max f_* + \varepsilon \max f_*' \cdot \frac{m}{1 - \varepsilon km}\right] < 1$$

The further estimates are analogous and the result is equivalent to (3.13).

In the case when $\omega = \text{const}, f \equiv 0$, estimates of the form (3.12) can be derived from the results of Bes'yes (given in /9/) dealing with periodic systems of standard type. In the notation used here, it gives

$$|\Phi_\varepsilon - \Psi| \leqslant \varepsilon \cdot 4\pi\omega^{-1}M(1 + \varepsilon\Delta t)\,e^{\varepsilon\Delta t} \tag{3.14}$$

where $\Delta = \max F'$. By virtue of the inequalities of the form $|\chi_0| \leqslant |\chi|$, $|\chi_*| \leqslant 2|\chi|$, we can conclude that $D \leqslant \Delta$, $m/\pi \leqslant 2M/\omega$, $km/\pi \leqslant 2\Delta/\omega$, and therefore the right-hand side of (3.11) is termwise better than (3.14):

$$2m \leqslant 4\pi M/\omega,\ mc = mkM \leqslant 2\pi\Delta M/\omega < 4\pi\Delta M/\omega,\ e^{\varepsilon Dt} \leqslant e^{\varepsilon\Delta t}$$

although it is obtained by relaxing (3.12).

4. The Chaplygin sledge (skate). Let a rigid body move along the Oxy plane. Its centre of mass S has coordinates x, y in fixed axes. A sharp blade is attached to the body. Its coordinates in the moving $S\xi\eta$ axes are r, d and it is directed along the $S\xi$ axis. Let φ be the angle between Ox and $S\xi$, m the mass of the body and ρ its central radius of inertia, and let elastic forces with potential $^1/_2k\,(x^2+y^2)$ act on the body. The kinetic energy and constraint equations have the form

$$T = {}^1/_2m\,(x^{\cdot 2} + y^{\cdot 2}) + {}^1/_2m\rho^2\varphi^{\cdot 2},\ -x^{\cdot}\sin\varphi + y^{\cdot}\cos\varphi + r\varphi^{\cdot} = 0$$

Let us assume that the dimensionally independent parameters $m = \rho = k = 1$, and after the change of variables (ξ, η are components of the vector OS in the $S\xi\eta$ system)

$$x = \xi\cos\varphi - \eta\sin\varphi,\ y = \xi\sin\varphi + \eta\cos\varphi$$

we will have

$$\begin{aligned}&\varphi^{\cdot} = \mu\eta^{\cdot},\ \mu(\xi) = -(r+\xi)^{-1},\ \mu^{\cdot} = \mu^2\xi^{\cdot},\ 1 + \mu\xi = -\mu r \\ &2L_* = \xi^{\cdot 2} + \eta^{\cdot 2} + 2\,(\xi\eta^{\cdot} - \eta\xi^{\cdot})\mu\eta^{\cdot} + (1 + \xi^2 + \eta^2)\mu^2\eta^{\cdot 2} - \\ &\quad \xi^2 - \eta^2\end{aligned} \tag{4.1}$$

while the relations (1.2) will become

$$p = p^K = \xi\eta^{\cdot} - \eta\xi^{\cdot} + (1 + \xi^2 + \eta^2)\mu\eta^{\cdot}\ (\mathrm{CT}),\ p = p^B = \text{const}\ (\mathrm{VT}). \tag{4.2}$$

Starting from (4.1) and following formulas (1.1) directly, leads to tedious manipulations. If we introduce the variables

$$p_\xi = \xi^{\cdot} - \eta\varphi^{\cdot},\ p_\eta = \eta^{\cdot} + \xi\varphi^{\cdot}$$

then the equations of motion will be reduced to an interesting form, which can allow of generalization to the whole class of Chaplygin systems:

$$\xi^{\cdot} = p_\xi - \eta p_\eta/r,\ \eta^{\cdot} = p_\eta + \xi p_\eta/r,\ \varphi^{\cdot} = -p_\eta/r \tag{4.3}$$

$$\begin{aligned}&p_\xi^{\cdot} + \xi - p_\eta^2/r = \mu^2\eta^{\cdot}\,(p - p^K)/r \\ &p_\eta^{\cdot} + \mu dp^K/dt + \eta + p_\xi p_\eta/r = -\mu^2\xi^{\cdot}\,(p - p^K)/r\end{aligned} \tag{4.4}$$

Here $p = p^K$ or p^B, and p^K participates without fail. It remains to solve the last equation for p_ξ; p_η (they also appear in $\mu dp^K/dt$).

Let us introduce the small parameter $\varepsilon = \rho/r$. When $\varepsilon \to 0$, the blade moves away from S, or the distribution of mass tends to a point distribution. The choice of $\rho = 1$ means, in fact, a change to dimensionless variables. In the limit, when $\varepsilon = 0$, we obtain $\varphi^{\cdot} = 0$, i.e. the constraint becomes integrable and we arrive at translational oscillations of the body. For small ε, Eqs. (4.3), (4.4) take the form $\varphi^{\cdot} = -\varepsilon p_\eta$ (the constraint) and

$$\begin{aligned}&\xi^{\cdot} = p_\xi - \varepsilon\eta p_\eta,\ \eta^{\cdot} = p_\eta + \varepsilon\xi p_\eta \\ &p_\xi^{\cdot} = -\xi - \varepsilon p_\eta^2,\ p_\eta^{\cdot} = -\eta + \varepsilon p_\xi p_\eta\end{aligned}$$

and in the limit (4.4) has the same form in the CT and VT (in the latter case it does not depend on the choice of initial conditions).

It is easy to show (e.g. by Bogolyubov reduction to standard form, followed by averaging) that there is no transgression to a first approximation. The main effect in the second approximation (the corresponding equations are not given here) will be the correction to the frequency of oscillations in η, while the oscillation frequency in ξ will remain unchanged (in the first case we have oscillations across the direction of the blade, and in the second case along the direction of the blade). In the CT the exact system (4.4), (4.3) is reduced to a quasilinear form and can be studied using the appropriate mthods.

5. A plate sliding on a blade. We shall demonstrate the possibility of representing a non-integrable constraint in the form of a perturbation in an integrable constraint, by artificial introduction of a small parameter in such a manner that a limit holonomic system does not exist.

A rigid body moves along a stationary Oyz plane. We denote the coordinates of its centre of mass by y, z and its angle of rotation by φ. We assume that a stationary blade directed along the Oy axis lies at the origin of coordinates. The the velocity of the body above it should be directed along the blade, and this yields the following constraint:

$$z^{\cdot} = y\varphi^{\cdot} \tag{5.1}$$

This represents a well-known example of an analytically simplest non-integrable constraint. The kinetic energy of the body is

$$T = {}^1/_2\,(y^{\cdot 2} + z^{\cdot 2}) + {}^1/_2\rho^2\varphi^{\cdot 2}$$

where $m = 1$, ρ is the mass of the body and central radius of inertia. We shall assume that the potential of the forces is $V(\varphi, y, z)$. Let us make the following change of variables:

$$\varphi = \alpha + \varepsilon x,\ \varepsilon = 1/\rho \tag{5.2}$$

Then the equations of motion (the transformed relation (5.1) is the first)

$$\begin{aligned}
&z^{\cdot} = \varepsilon y x^{\cdot},\ y^{\cdot\cdot} - \varepsilon^2 y x^{\cdot 2} + \partial V/\partial y = -\varepsilon p_z x^{\cdot} \\
&(1 + \varepsilon^2 y^2) x^{\cdot\cdot} + 2\varepsilon^2 y y^{\cdot} x^{\cdot} + \varepsilon \partial V/\partial \varphi + \varepsilon \partial V/\partial z \cdot y = \varepsilon p_z y^{\cdot} \\
&p_z = z^{\cdot} = \varepsilon y x^{\cdot}\ (\mathrm{CT})\,,\ p_z^{\cdot} = -\partial V/\partial z\quad (\mathrm{VT})
\end{aligned}$$

will have a form suitable for use with the theory developed here (we must substitute the expression (5.2) into $\varepsilon\partial V/\partial\varphi$). When $\varepsilon = 0$, and we obtain formally a system in the x, y plane: a point of unit mass moves in the field of potential $W(y) = V(\alpha, y, z_0)$.

The integrals of motion: the total energy

$$H = {}^1/_2\,((1 + \varepsilon^2 y^2) x^{\cdot 2} + y^{\cdot 2}) + V$$

is always conserved. When $\partial V/\partial\varphi \equiv 0$, the VT system has the integral

$$(1 + \varepsilon^2 y^2) x^{\cdot} - \varepsilon\, p_z y = k$$

and the CT system, when $\partial V/\partial\varphi \equiv 0$ and the potential $V(y, z)$ is axisymmetric, has a similar integral (of the momentum relative to the origin of coordinates)

$$(1 + \varepsilon^2 y^2) x^{\cdot} - \varepsilon z y^{\cdot} = k$$

If $V = V_1(y) + V_2(x, z)$, then the variables can be separated in the classical system and the energy integral will decompose into two integrals:

$$\begin{aligned}
&{}^1/_2\,(1 + \varepsilon^2 y^2) x^{\cdot 2} + V_2(x, z) = K \\
&{}^1/_2 y^{\cdot 2} + V_1(y) = h
\end{aligned}$$

Let us consider inertial motion with initial conditions

$$\varphi_0^{\cdot} = \varepsilon u,\ y_0^{\cdot} = v,\ z_0^{\cdot} = 0,\ \varphi_0 = y_0 = z_0 = 0$$

From the integrals of motion we find that in the case of CT

$$\begin{aligned}
&y^{\cdot} = v,\ x^{\cdot} = u\,(1 + \varepsilon^2 y^2)^{-1/2},\ z^{\cdot} = \varepsilon u y\,(1 + \varepsilon^2 y^2)^{-1/2} \\
&y = vt,\ x = u\,(\varepsilon v)^{-1}\,\mathrm{Arsh}\,\varepsilon y,\ z = u\,(\varepsilon v)^{-1}\,(\sqrt{1 + \varepsilon^2 y^2} - 1)
\end{aligned}$$

When VT is used, we have $p_z^{\cdot} = 0$, and we can therefore assume that when $p_z = 0$:

$$y^{\cdot} = \sqrt{v^2 + \frac{\varepsilon^2 y^2 u^2}{1 + \varepsilon^2 y^2}},\quad x^{\cdot} = \frac{u}{1 + \varepsilon^2 y^2},\quad z^{\cdot} = \frac{\varepsilon u y}{1 + \varepsilon^2 y^2}$$

The effects of the CT and VT are identical apart from terms of the order of ε^2. The loss of accuracy caused by neglecting terms of order ε^2 is readily apparent. A qualitative description in this case is simple.

6. Conclusion. We know that stationary non-integrable constraints are possible only in the case when the number of defining coordinates is not less than three. Above we have considered systems for which this minimum was attained. When the degrees of freedom are insufficient, the most unavoidable deviations seem to appear in the CT and VT effects. In the first-order approximation the quantitative (but not qualitative) deviations appeared in the case of non-stationary constraints, while in the case of stationary constraints the deviations should obviously only be expected in the higher-order approximations.

It can be shown that in weakly non-holonomic Chaplygin systems with linear stationary constraints, the secular VT and CT effects will not appear in any approximation whatsoever. Indeed, the equations of motion with constrains of the type (1.4) are

$$\frac{d}{dt}\frac{\partial T^*}{\partial x_\lambda^{\cdot}} - \frac{\partial T^*}{\partial x_\lambda} = \varepsilon \sum_s p_s \left(\frac{d}{dt}\frac{\partial f_s}{\partial x_\lambda^{\cdot}} - \frac{\partial f_s}{\partial x_\lambda}\right)$$

In the CT model $p_s = \partial T/\partial x_s^{\cdot}$ and the right-hand sides obtained are quadratic with respect to the velocities. A reversible system appears for which Moser's theorem /12/ on conserving the tori in conditionally periodic motions holds (assuming non-degeneracy when $\varepsilon \to 0$). The VT model in which $p_s = \mathrm{const}$ yields, simply, for all initial conditions, a Hamiltonian perturbation of the gyroscopic-force type.

The author thanks V.V. Rumyantsev, A.V. Karapetyan, V.V. Kozlov, A.I. Neishtadt and A.S. Sumbstov for valuable suggestions, and V.F. Zhuravlev for useful comments.

REFERENCES

1. HERTZ H., Die Prinziepien der Mechanik in neuem Zusammenhänge dargestellt: Ges. Werke. B. 3 Leipzig: Barth, 1894.
2. APPEL' P., Theoretical Mechanics, 2, Moscow, Fizmatgiz, 1960.
3. SYNGE J.L., Tensor Methods in Dynamics, /Russian translation/, Moscow, Gostekhizdat, 1947.
4. CARATHÉODORY C., Der Schlitten, Z. angew. Math. und Mech. 13, 2, 1933.
5. NEIMARK YU.I. and FUFAYEV N.A., Dynamics of Non-holonomic Systems. Moscow, Nauka, 1967.
6. KARAPETYAN A.V., On realizing non-holonomic constraints by forces of viscous friction and stability of Celtic stones. PMM, 45, 1, 1981.
7. KOZLOV V.V., Dynamics of systems with non-integrable constraints. III Vestn. MGU, Ser.1, Matematika, mekhanika, 3, 1983.
8. KOZLOV V.V., Realization of non-integrable constraints in classical mechanics.Dokl. A ad. Nauk SSSR, 272, 3, 1983.
9. ZHURAVLEV V.F. and KLIMOV D.M., The Solid State Wave Gyroscope. Moscow, Nauka, 1985.
10. ARNOL'D V.I., Additional Chapters in the Theory of Ordinary Differential Equations. Moscow, Nauka, 1978.
11. TATARINOV YA.V., Lectures on Classical Dynamics, Moscow, Izd-vo MGU, 1984.
12. MOZER YU., On expanding conditionally periodic motions in convergent power series. Uspekhi Mat. Nauk, 24, 2, 1969.

Translated by L.K.

PMM U.S.S.R.,Vol.51,No.5,pp.586-592,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON THE CHANGE IN THE ADIABATIC INVARIANT ON CROSSING A SEPARATRIX IN SYSTEMS WITH TWO DEGREES OF FREEDOM*

A.I. NEISHTADT

Hamiltonian systems with two degrees of freedom are studied. One degree of freedom corresponds to rapid motion, and the other to slow motion. The phase point intersects the separatrix of the rapid motion. Formulas are obtained for the change in the adiabatic invariant during this crossing. An example is solved, dealing with the change in the adiabatic invariant of an asteroid near the 3:1 resonance with Jupiter.

1. Formulation of the problem. A number of problems of the theory of oscillations lead to Hamiltonian systems with a Hamiltonian of the form $H = H(p, q, y, x)$, where q, $\varepsilon^{-1}x$ are the coordinates, p, y the associated moments, $\varepsilon > 0$ is a small parameter and $H \in C^\infty$. The variables p, q will be called rapid, and y, x slow. The Hamiltonian system for p, q with $(y, x) = \text{const}$ will be called rapid or unperturbed. The Hamiltonian of the type shown characterizes, e.g. the motion of an asteroid in the bounded three-body problem near a resonance.

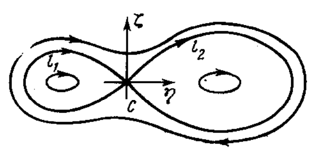

Fig.1

Below we assume that the phase plane of the rapid system contains the separatrices shown in Fig.1 for all values of the slow variables under cosideration. When the slow variables are varied, the phase point intersects the separatrix. The motion away from the sepatrix is characterized by a quantity which is preserved with a high degree of accuracy, namely the adiabatic invariant (AI) /1/. The neighbourhood of the sepratrices

*Prikl.Matem.Mekhan.,51,5,750-757,1987